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Stirling permutation codes

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A B S T R A C T

The development of the theory of the second-order Eulerian polynomials began with the works of Buckholtz and Carlitz in their studies of an asymptotic expansion. Gessel-Stanley introduced Stirling permutations and provided combinatorial interpretations for the second-order Eulerian polynomials in terms of Stirling permutations. The Stirling permutations have been extensively studied by many researchers. The motivation of this paper is to develop a general method for finding equidistributed statistics on Stirling permutations. Firstly, we show that the up-down-pair statistic is equidistributed with the ascent-plateau statistic, and that the exterior up-down-pair statistic is equidistributed with the left ascent-plateau statistic. Secondly, we introduce the Stirling permutation code (called SP-code). A large number of equidistribution results follow from simple applications of the SP-codes. In particular, we find that six bivariable set-valued statistics are equidistributed on the set of Stirling permutations, and we generalize a classical result on trivariate version of the second-order Eulerian polynomial, which was independently established by Dumont and Bóna. Thirdly, we explore the bijections among Stirling permutation codes, perfect matchings and trapezoidal words. We then show the e-positivity of the enumerators of Stirling permutations by left ascent-plateaux, exterior up-down-pairs and right plateau-descents. In the final part, the

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e-positivity of the multivariate k-th order Eulerian polynomials is established, which improves a classical result of Janson-Kuba-Panholzer and generalizes a recent result of Chen-Fu. These e-positive expansions are derived from the combinatorial theory of context-free grammars.

Contents

1. Introduction

1.1. Notation and preliminaries

The development of the theory of the second-order Eulerian polynomials began with the works of Buckholtz [3] and Carlitz [4] in their studies of an asymptotic expansion. Further developments continued with the contributions of Riordan [30], Gessel-Stanley [15], Dumont [10], Park [32], Bóna [1], Janson-Kuba-Panholzer [19], Haglund-Visontai [16] and Chen-Fu [7,8]. The aim of this paper is to give original and substantial generalizations of these polynomials.

For each positive integer n and each complex number x, one can define $S_n(x)$ by the equation

$$\mathrm{e}^{nx} = \sum_{r=0}^{n} \frac{(nx)^r}{r!} + \frac{(nx)^n}{n!} S_n(x). \tag{1}$$

The study of (1) was initiated by Ramanujan [29], where he made the assertion (in a different notation) that

$$S_n(1) = \frac{n!}{2}\left(\frac{\mathrm{e}}{n}\right)^n - \frac{2}{3} + \frac{4}{135n} + O(n^{-2}),$$

which was independently proved in 1928 by Szegö and Watson.

Buckholtz [3] found that $S_n(x) = \sum_{r=0}^{k-1} \frac{1}{n^r} U_r(x) + O(n^{-k})$, where

$$U_r(x) = (-1)^r \left(\frac{x}{1-x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^r \frac{x}{1-x} = (-1)^r \frac{C_r(x)}{(1-x)^{2r+1}},$$

and $C_r(x)$ is a polynomial of degree r. Carlitz [4] discovered that

$$C_n(x) = (1-x)^{2n+1} \sum_{k=0}^{\infty} \left\{ {n+k \atop k} \right\} x^k,$$

where $\left\{ {n \atop k} \right\}$ are the *Stirling numbers of the second kind*. The polynomials $C_n(x)$ are now known as the *second-order Eulerian polynomials* and they satisfy the following recursion

$$C_{n+1}(x) = (2n+1)xC_n(x) + x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}C_n(x),\ C_0(x) = 1.$$

In particular, $C_1(x) = x, C_2(x) = x + 2x^2, C_3(x) = x + 8x^2 + 6x^3$. In [30], Riordan found that $C_n(x)$ are the enumerators of Riordan trapezoidal words of length n by number of distinct numbers. Subsequently, Gessel-Stanley [15] discovered that $C_n(x)$ are the descent polynomials for Stirling permutations of the multiset $[n]_2 = \{1, 1, 2, 2, \ldots, n, n\}$. The Stirling permutations have been extensively studied in the past decades, see [8,12,13,21,22,26,32,33] and references therein.

For $\mathbf{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{N}^n$, let $\mathbf{n} = \{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ be a multiset, where the element i appears m_i times. A *multipermutation* of $\mathbf{n}$ is a sequence of its elements. Denote by $\mathfrak{S}_{\mathbf{n}}$ the set of multipermutations of $\mathbf{n}$. We say that the multipermutation σ of $\mathbf{n}$ is a *Stirling permutation* if $\sigma_s \geqslant \sigma_i$ as soon as $\sigma_i = \sigma_j$ and $i < s < j$. Denote by $\mathcal{Q}_{\mathbf{n}}$ the set of Stirling permutations of $\mathbf{n}$. When $m_1 = \cdots = m_n = 1$, the set $\mathcal{Q}_{\mathbf{n}}$ reduces to the symmetric group $\mathfrak{S}_n$, which is the set of all permutations of the set $[n] = \{1, 2, \ldots, n\}$. When $m_1 = \cdots = m_n = 2$, the set $\mathcal{Q}_{\mathbf{n}}$ reduces to $\mathcal{Q}_n$, which is the set of all Stirling permutations of the multiset $[n]_2$. Except where explicitly stated, we always assume that all Stirling permutations belong to $\mathcal{Q}_n$, and for $\sigma \in \mathcal{Q}_n$, we set $\sigma_0 = \sigma_{2n+1} = 0$. For example, $\mathcal{Q}_1 = \{11\},\ \mathcal{Q}_2 = \{1122, 1221, 2211\}$.

Definition 1. For $\sigma \in \mathfrak{S}_{\mathbf{n}}$, any entry σ_i is called

- (i) an *ascent* (resp. *descent*, *plateau*) if $\sigma_i < \sigma_{i+1}$ (resp. $\sigma_i > \sigma_{i+1}$, $\sigma_i = \sigma_{i+1}$), where $i \in \{0, 1, 2, \ldots, m_1 + m_2 + \cdots + m_n\}$ and we set $\sigma_0 = \sigma_{m_1+m_2+\cdots+m_n+1} = 0$, see [1,15];
- (ii) an *ascent-plateau* (resp. *plateau-descent*) if $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$ (resp. $\sigma_{i-1} = \sigma_i > \sigma_{i+1}$), where $i \in \{2, 3, \ldots, m_1 + m_2 + \cdots + m_n - 1\}$, see [23,25];

(*iii*) a *left ascent-plateau* if $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$, where $i \in \{1, 2, 3, \ldots, m_1 + m_2 + \cdots + m_n - 1\}$ and we set $\sigma_0 = 0$, see [23,25];

(*iv*) a *right plateau-descent* if $\sigma_{i-1} = \sigma_i > \sigma_{i+1}$, where $i \in \{2, 3, \ldots, m_1+m_2+\cdots+m_n\}$ and we set $\sigma_{m_1+m_2+\cdots+m_n+1} = 0$, see [22,26].

Let $\operatorname{asc}(\sigma)$ (resp. $\operatorname{des}(\sigma)$, $\operatorname{plat}(\sigma)$, $\operatorname{ap}(\sigma)$, $\operatorname{pd}(\sigma)$, $\operatorname{lap}(\sigma)$, $\operatorname{rpd}(\sigma)$) denotes the number of ascents (resp. descents, plateaux, ascent-plateaux, plateau-descents, left ascent-plateaux, right plateau-descents) of σ. The reverse bijection $\sigma \to \sigma^r$ on $\mathcal{Q}_n$ defined by $\sigma_i^r = \sigma_{2n+1-i}$ shows that

$$\sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{asc}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{des}(\sigma)}, \quad \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{ap}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{pd}(\sigma)}, \quad \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{lap}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{rpd}(\sigma)}.$$

In [1], Bóna introduced the plateau statistic plat and discovered that $C_n(x) = \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{plat}(\sigma)}$, which leads to a remarkable equidistributed result:

$$\sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{asc}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{plat}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{des}(\sigma)}. \tag{2}$$

It should be noted that the plateau statistic has been considered by Dumont [10] in the name of the repetition statistic, and it went unnoticed until it was independently studied by Bóna. A trivariate version of the second-order Eulerian polynomial is defined by

$$C_n(x,y,z) = \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{asc}(\sigma)} y^{\operatorname{des}(\sigma)} z^{\operatorname{plat}(\sigma)}. \tag{3}$$

Clearly, $C_1(x,y,z) = xyz$. Dumont [10, p. 317] found that

$$C_{n+1}(x,y,z) = xyz\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) C_n(x,y,z), \tag{4}$$

which implies that $C_n(x,y,z)$ is symmetric in the variables x, y and z, and so (2) holds. The symmetry of $C_n(x,y,z)$ was rediscovered by Janson [18, Theorem 2.1] by constructing an urn model. In [16], Haglund-Visontai introduced a refinement of the polynomial $C_n(x,y,z)$ by indexing each ascent, descent and plateau by the values where they appear. Using the theory of context-free grammars, Chen-Fu [8] found that $C_n(x,y,z)$ is e-positive, i.e.,

$$C_n(x,y,z) = \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k} (x+y+z)^i (xy+yz+zx)^j (xyz)^k, \tag{5}$$

where the coefficient $\gamma_{n,i,j,k}$ equals the number of 0-1-2-3 increasing plane trees on $[n]$ with k leaves, j vertices with degree one and i degree two vertices.

A *rooted tree* of order n with the vertices labeled $1, 2, \ldots, n$, is an increasing tree if the node labeled 1 is distinguished as the root, and the labels along any path from the

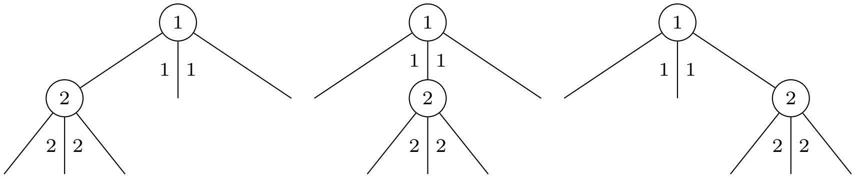

Fig. 1. The ternary increasing trees of order 2 encoded by 2211, 1221, 1122, and their SP-codes are given by $((0,0),(1,1)),((0,0)(1,2))$ and $((0,0)(1,3))$, respectively.

root are increasing. An *increasing plane tree*, usually called *plane recursive tree*, is an increasing tree with the children of each vertex are linearly ordered (from left to right, say). A *0-1-2-$\cdots$-k increasing plane tree* on $[n]$ is an increasing plane tree, where each vertex has at most k children. The *degree* of a vertex in a rooted tree is meant to be the number of its children (sometimes called *outdegree*). The *depth-first walk* of a rooted plane tree starts at the root, goes first to the leftmost child of the root, explores that branch (recursively, using the same rules), returns to the root, and continues with the next child of the root, until there are no more children left.

The following definition will be used repeatedly.

Definition 2 *([10]).* A *ternary increasing tree* of size n is an increasing plane tree with $3n+1$ nodes in which each interior node has a label and three children (a left child, a middle child and a right child), and exterior nodes have no children and no labels.

Let $\mathcal{T}_n$ denote the set of ternary increasing trees of size n, see Fig. 1 for instance. For any $T \in \mathcal{T}_n$, it is clear that T has exactly $2n+1$ exterior nodes. Let $\operatorname{exl}(T)$ (resp. $\operatorname{exm}(T)$, $\operatorname{exr}(T)$) denotes the number of exterior left nodes (resp. exterior middle nodes, exterior right nodes) in T. Using a recurrence relation that is equivalent to (4), Dumont [10, Proposition 1] found that

$$C_n(x,y,z)=\sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{asc}(\sigma)}y^{\operatorname{plat}(\sigma)}z^{\operatorname{des}(\sigma)}=\sum_{T\in\mathcal{T}_n} x^{\operatorname{exl}(T)}y^{\operatorname{exm}(T)}z^{\operatorname{exr}(T)}. \tag{6}$$

1.2. Motivation and the organization of the paper

A bijection between plane recursive trees and Stirling permutations was independently found by Koganov [20] and Janson [18]. Subsequently, Janson-Kuba-Panholzer [19, Section 3] showed that this bijection naturally extends to a bijection between $(k+1)$-ary increasing trees and k-Stirling permutations, which was independently introduced by Gessel [32, p46]. By taking $k=2$ in the proof of [19, Theorem 1], the bijection ϕ between ternary increasing trees and Stirling permutations can be described as follows (we give a detail description of it for convenience):

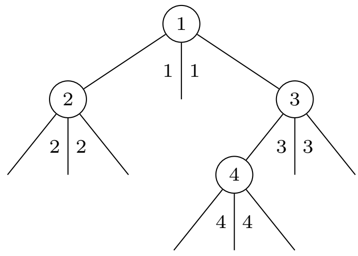

Fig. 2. An order 4 ternary increasing tree encoded by 22114433, and its SP-code is $((0,0),(1,1),(1,3),(3,1))$.

(i) Given $T \in \mathcal{T}_n$. Between the 3 edges of T going out from a node labeled v, we place 2 integers v. Now we perform the depth-first walk and code T by the sequence of the labels visited as we go around T. Let $\phi(T)$ be the code. In particular, the ternary increasing tree of order 1 is encoded by the Stirling permutation 11. A ternary increasing tree of order n is encoded by a string of $2n$ integers, where each of the labels $1, 2, \ldots, n$ appears exactly 2 times. Clearly, the code $\phi(T)$ is a Stirling permutation, see Fig. 2 for illustration;

(ii) The inverse of ϕ can be described as follows. Given $\sigma \in \mathcal{Q}_n$. We proceed recursively starting at step one by decomposing σ as $u_1 1 u_2 1 u_3$, where the u_i's are again Stirling permutations. The smallest label in each u_i is attached to the root node labeled 1. One can recursively apply this procedure to each u_i to obtain the tree representation, and $\phi^{-1}(\sigma)$ is a ternary increasing tree.

Based on the work of Bóna [1], Chen-Fu [8], Dumont [10], Haglund-Visontai [16], Gessel-Stanely [15] and Janson-Kuba-Panholzer [19], this paper is devoted to the following problem.

Problem 3. Develop a general method for finding equidistributed statistics on $\mathcal{Q}_n$.

In Section 2, we introduce the up-down-pair statistic ud and the exterior up-down-pair statistic eud on Stirling permutations, and we show that ud is equidistributed with ap and eud is equidistributed with lap. Therefore, we get

$$\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{ap}\,(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{ud}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{pd}(\sigma)},$$

$$\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{eud}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{rpd}(\sigma)}. \tag{7}$$

In Section 3, we first introduce an encoding of Stirling permutation (called SP-code), and we then present various results concerning Problem 3. In particular, in Theorems 13 and 14, we present bivariate generalizations of (2). The last two identities given in Theorem 14 generalize (2) and (7) simultaneously. In Section 4, we establish bijections among

SP-codes, trapezoidal words and perfect matchings. In Section 5, we show the e-positivity of the enumerators of Stirling permutations by (lap, eud, rpd). In Section 6, we introduce the multivariate k-th order Eulerian polynomial, and we find that it is e-positive, which generalizes (5) and improves a classical result of Janson-Kuba-Panholzer [19].

2. The ascent-plateau and up-down-pair statistics

The number of elements in a set C is called the *cardinality* of C, written as $\#C$. The *type A Eulerian polynomials* $A_n(x)$ [17], the *type B Eulerian polynomials* $B_n(x)$ [2], the *ascent-plateau polynomials* (also called 1/2-*Eulerian polynomials*) $M_n(x)$ [23,31] and the *left ascent-plateau polynomials* $N_n(x)$ [23] can be respectively defined as follows:

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}\,(\pi)},\ B_n(x) = \sum_{\pi \in \mathcal{B}_n} x^{\mathrm{des}_B(\pi)},$$
$$M_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{ap}\,(\sigma)},\ N_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)},$$

where $\mathcal{B}_n$ denotes the hyperoctahedral group of rank n,

$$\mathrm{des}_B(\pi) = \#\{i \in \{0,1,2,\ldots,n-1\} \mid \pi(i) > \pi(i+1)\},\ \text{where we set } \pi(0) = 0.$$

These polynomials share several similar properties, including recursions [15,23,28], real-rootedness [1,16,28], combinatorial expansions [8,21,26,34] and asymptotic distributions [17]. Here we collect the recursions of these polynomials:

$$A_{n+1}(x) = (n+1)xA_n(x) + x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}A_n(x),$$
$$B_{n+1}(x) = (2nx+1+x)B_n(x) + 2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}B_n(x),$$
$$M_{n+1}(x) = (2nx+1)M_n(x) + 2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}M_n(x),$$
$$N_{n+1}(x) = (2n+1)xN_n(x) + 2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}N_n(x),$$

with $A_0(x) = B_0(x) = M_0(x) = N_0(x) = 1$. There are close connections among these polynomials (see [24] for details). According to [24, Proposition 1], we have

$$2^n A_n(x) = \sum_{i=0}^{n} \binom{n}{i} N_i(x)N_{n-i}(x),\ B_n(x) = \sum_{i=0}^{n} \binom{n}{i} M_i(x)N_{n-i}(x). \tag{8}$$

Let $\mathcal{Q}_n^{(1)}$ be the set of Stirling permutations of the multiset $\{1, 2^2, 3^2, \ldots, n^2, (n+1)^2\}$, i.e., this multiset has exactly one 1 and two copies of i for each $2 \leqslant i \leqslant n+1$. In particular,

$$\mathcal{Q}_2^{(1)} = \{12233, 12332, 13322, 33122, 22133, 22331, 23321, 33221\}.$$

By considering the position of the entry 1 in a Stirling permutation $\sigma \in \mathcal{Q}_n^{(1)}$, the following result immediately follows from (8).

Proposition 4. *For $n \geqslant 1$, we have*

$$2^n A_n(x) = \sum_{\sigma \in \mathcal{Q}_n^{(1)}} x^{\mathrm{lap}\,(\sigma)},\ B_n(x) = \sum_{\sigma \in \mathcal{Q}_n^{(1)}} x^{\mathrm{ap}\,(\sigma)}.$$

Definition 5. Let $\sigma \in \mathcal{Q}_n$. An entry σ_i is called an up-down-pair entry if $\sigma_{i-1} < \sigma_i = \sigma_j > \sigma_{j+1}$, where $i < j$. The two equal entries σ_i and σ_j may appear arbitrarily far apart. The up-down-pair statistic ud and the exterior up-down-pair statistic eud are respectively defined as follows:

$$\mathrm{ud}(\sigma) = \#\{i \in [2n-2] :\ \sigma_i \text{ is an up-down-pair entry, where we set } \sigma_0 = 0\},$$
$$\mathrm{eud}(\sigma) = \#\{i \in [2n-1] :\ \sigma_i \text{ is an up-down-pair entry, where we set } \sigma_0 = \sigma_{2n+1} = 0\}.$$

Example 6. We have

$$\mathrm{ud}(123321) = \mathrm{ud}(0123321) = 2, \mathrm{ud}(331221) = \mathrm{ud}(0331221) = 2,$$
$$\mathrm{eud}(123321) = \mathrm{eud}(01233210) = 3, \mathrm{eud}(331221) = \mathrm{eud}(03312210) = 2.$$

The main result of this section is given as follows.

Theorem 7. *For $n \geqslant 1$, we have*

$$\sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{ap}\,(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{ud}(\sigma)},\ \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{eud}(\sigma)}. \tag{9}$$

Proof. Let

$$M_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{ap}\,(\sigma)} = \sum_{i=0}^{n-1} M_{n,i}x^i,\ N_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)} = \sum_{i=1}^{n} N_{n,i}x^i.$$

Then the coefficients $M_{n,i}$ and $N_{n,i}$ respectively satisfy the recurrence relations

$$\begin{aligned} M_{n+1,i} &= (2i+1)M_{n,i} + (2n-2i+2)M_{n,i-1}, \\ N_{n+1,i} &= 2iN_{n,i} + (2n-2i+3)N_{n,i-1}, \end{aligned} \tag{10}$$

with the initial conditions $M_{0,0} = N_{0,0} = 1$ and $M_{0,i} = N_{0,i} = 0$ if $i > 0$, see [23].

Let $m_{n,i} = \#\{\sigma \in \mathcal{Q}_n : \mathrm{ud}(\sigma) = i\}$. It is clear that $m_{1,0} = M_{1,0} = 1$, since $\mathrm{ud}(11) = \mathrm{ud}(011) = 0$. There are two ways to obtain an element $\sigma' \in \mathcal{Q}_{n+1}$ with $\mathrm{ud}(\sigma') = i$ from an element $\sigma \in \mathcal{Q}_n$ by inserting two copies of n into consecutive positions:

(c_1) If $\mathrm{ud}(\sigma) = i$, then we can insert the two copies of n before an up-down-pair entry or right after the second appearance of it. Moreover, we can insert the two copies of n at the end of σ. This accounts for $(2i + 1)m_{n,i}$ possibilities;

(c_2) If $\mathrm{ud}(\sigma) = i - 1$, then we insert the two copies of n into one of the remaining $2n + 1 - (2(i - 1) + 1) = 2n - 2i + 2$ positions. This accounts for $(2n - 2i + 2)m_{n,i-1}$ possibilities.

Thus $m_{n,i}$ satisfy the same recursion and initial conditions as $M_{n,i}$, so they agree.

Define $u_{n,i} = \#\{\sigma \in \mathcal{Q}_n : \mathrm{eud}(\sigma) = i\}$. Clearly, $u_{1,1} = N_{1,1} = 1$, since $\mathrm{eud}(0110) = 1$. Similarly, there are two ways to obtain an element $\sigma' \in \mathcal{Q}_{n+1}$ with $\mathrm{eud}(\sigma') = i$ from an element $\sigma \in \mathcal{Q}_n$ by inserting two copies of n into consecutive positions:

(c_1) If $\mathrm{eud}(\sigma) = i$, then we can insert the two copies of n before an up-down-pair entry or right after the second appearance of it. This accounts for $2iu_{n,i}$ possibilities;

(c_2) If $\mathrm{eud}(\sigma) = i - 1$, then we insert the two consecutive copies of n into one of the remaining $2n+1-2(i-1) = 2n-2i+3$ positions. This accounts for $(2n-2i+3)u_{n,i-1}$ possibilities.

Thus $u_{n,i}$ satisfy the same recursion and initial conditions as $N_{n,i}$, so they agree. □

3. Problem 3 and the Stirling permutation code

Recall that a sequence $(e_1, e_2, \ldots, e_n)$ is an *inversion sequence* if $0 \leqslant e_i < i$ for all $i \in [n]$. It is well known that inversion sequences of length n are in bijection with permutations in $\mathfrak{S}_n$. As a dual of inversion sequence, by using the bijection ϕ (see subsection 1.2 for details), we shall introduce a common code for ternary increasing trees and Stirling permutations.

Recall that for any ternary increasing tree $T \in \mathcal{T}_n$, each interior node has a label and three children (a child at the left, a middle child and a right child), and the exterior nodes have no children and no labels. For convenience, we introduce the following definition.

Definition 8. A simplified ternary increasing tree is a ternary increasing tree with no exterior nodes. The *degree* of a vertex in a ternary increasing tree is meant to be the number of its children in the simplified ternary increasing tree.

In fact, a simplified ternary increasing tree is the same as the ordinary ternary increasing tree, it is only a simplified version. A node in a simplified ternary increasing tree with no children is called a *leaf*, and any interior node has at most three children (a left child, a middle child or a right child). For example, Fig. 3 gives the set of simplified ternary increasing trees of order 2. In the sequel, a ternary increasing tree is always meant to be a simplified ternary increasing tree. A ternary increasing tree of size n can be built up from the root 1 by successively adding nodes $2, 3, \ldots, n$. Clearly, node 2 is a

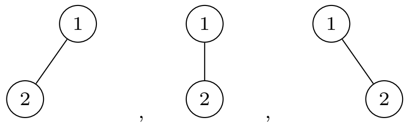

Fig. 3. The simplified ternary increasing trees of order 2.

child of the root 1 and the root 1 has at most three children, see Fig. 3 for instance. For $2 \leqslant i \leqslant n$, when node i is inserted, we distinguish three cases:

(c_1) if it is the left child of a node $v \in [i-1]$, then the node i is coded as $[v, 1]$;
(c_2) if it is the middle child of a node $v \in [i-1]$, then the node i is coded as $[v, 2]$;
(c_3) if it is the right child of a node $v \in [i-1]$, then the node i is coded as $[v, 3]$.

Thus the node i is coded as a 2-tuple (a_{i-1}, b_{i-1}), where $1 \leqslant a_{i-1} \leqslant i-1$, $1 \leqslant b_{i-1} \leqslant 3$ and $(a_i, b_i) \neq (a_j, b_j)$ for all $1 \leqslant i < j \leqslant n-1$. By convention, the root 1 is coded as $(0, 0)$. Therefore, a ternary increasing tree of size n corresponds naturally to a *build-tree code* $((0, 0), (a_1, b_1), \ldots, (a_{n-1}, b_{n-1}))$. Using the bijection ϕ between ternary increasing trees and Stirling permutations, one can see that the build-tree code is the same as the *Stirling permutation code*, which is defined as follows.

Definition 9. A 2-tuples sequence $C_n = ((0, 0), (a_1, b_1), (a_2, b_2) \ldots, (a_{n-1}, b_{n-1}))$ of length n is called a Stirling permutation code (SP-code for short) if $1 \leqslant a_i \leqslant i$, $1 \leqslant b_i \leqslant 3$ and $(a_i, b_i) \neq (a_j, b_j)$ for all $1 \leqslant i < j \leqslant n-1$.

Let CQ_n be the set of SP-codes of length n. In particular, $\mathrm{CQ}_1 = \{(0, 0)\}$ and $\mathrm{CQ}_2 = \{(0, 0)(1, 1),\ (0, 0)(1, 2),\ (0, 0)(1, 3)\}$, see Fig. 1.

Theorem 10. *The set* CQ_n *is in a natural bijection with the set* $\mathcal{Q}_n$, *i.e.*, $\mathrm{CQ}_n \cong \mathcal{Q}_n$.

Proof. For any $n \geqslant 2$, there are three cases to obtain an element of $\mathcal{Q}_n$ from an element $\sigma \in \mathcal{Q}_{n-1}$ by putting the two copies of n between σ_i and σ_{i+1}: $\sigma_i < \sigma_{i+1}$, $\sigma_i = \sigma_{i+1}$, $\sigma_i > \sigma_{i+1}$. Set $\Gamma(11) = (0, 0)$. When $n \geqslant 2$, the bijection $\Gamma : \mathcal{Q}_n \to \mathrm{CQ}_n$ can be defined as follows:

(c_1) $\sigma_i < \sigma_{i+1}$ if and only if $(a_{n-1}, b_{n-1}) = (\sigma_{i+1}, 1)$;
(c_2) $\sigma_i = \sigma_{i+1}$ if and only if $(a_{n-1}, b_{n-1}) = (\sigma_{i+1}, 2)$;
(c_3) $\sigma_i > \sigma_{i+1}$ if and only if $(a_{n-1}, b_{n-1}) = (\sigma_i, 3)$. □

Example 11. Given $\sigma = 551443312662 \in \mathcal{Q}_6$. We give the procedure of creating its SP-code:

$$11 \Leftrightarrow (0, 0),$$

$$\begin{aligned}
1122 &\Leftrightarrow (0,0)(1,3),\\
133122 &\Leftrightarrow (0,0)(1,3)(1,2),\\
14433122 &\Leftrightarrow (0,0)(1,3)(1,2)(3,1),\\
5514433122 &\Leftrightarrow (0,0)(1,3)(1,2)(3,1)(1,1),\\
551443312662 &\Leftrightarrow (0,0)(1,3)(1,2)(3,1)(1,1)(2,2).
\end{aligned}$$

Thus $\Gamma(\sigma) = (0,0)(1,3)(1,2)(3,1)(1,1)(2,2)$. Conversely, we get $\Gamma^{-1}(\Gamma(\sigma)) = \sigma$.

For $\sigma \in \mathcal{Q}_n$, let

$$\begin{aligned}
\mathrm{Asc}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} < \sigma_i\},\\
\mathrm{Plat}(\sigma) &= \{\sigma_i \mid \sigma_i = \sigma_{i+1}\},\\
\mathrm{Des}(\sigma) &= \{\sigma_i \mid \sigma_i > \sigma_{i+1}\},\\
\mathrm{Lap}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} < \sigma_i = \sigma_{i+1}\},\\
\mathrm{Rpd}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} = \sigma_i > \sigma_{i+1}\},\\
\mathrm{Eud}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} < \sigma_i = \sigma_j > \sigma_{j+1},\ i < j\},\\
\mathrm{Dasc}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} < \sigma_i < \sigma_{i+1}\},\\
\mathrm{Dplat}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} > \sigma_i = \sigma_{i+1}\},\\
\mathrm{Ddes}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} > \sigma_i > \sigma_{i+1}\},\\
\mathrm{Pasc}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} = \sigma_i < \sigma_{i+1}\},\\
\mathrm{Apd}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} < \sigma_i = \sigma_{i+1} > \sigma_{i+1}\},\\
\mathrm{Uu}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} < \sigma_i = \sigma_j < \sigma_{j+1},\ i < j\},\\
\mathrm{Dd}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} > \sigma_i = \sigma_j > \sigma_{j+1},\ i < j\}
\end{aligned}$$

denote the sets of ascents, plateaux, descents, left ascent-plateaux, right plateau-descents, exterior up-down-pairs, double ascents, descent-plateaux, double descents, plateau-ascents, ascent-plateau-descents, up-up-pairs and down-down-pairs of σ, respectively. We use $\mathrm{dasc}\,(\sigma)$, $\mathrm{dplat}(\sigma)$, $\mathrm{ddes}\,(\sigma)$, $\mathrm{pasc}(\sigma)$, $\mathrm{apd}(\sigma)$, $\mathrm{uu}(\sigma)$ and $\mathrm{dd}(\sigma)$ to respectively denote the number of double ascents, descent-plateaux, double descents, plateau-ascents, ascent-plateau-descents, up-up-pairs and down-down-pairs of σ, i.e., $\mathrm{dasc}\,(\sigma) = \#\mathrm{Dasc}(\sigma)$, $\mathrm{dplat}(\sigma) = \#\mathrm{Dplat}(\sigma)$, $\mathrm{ddes}\,(\sigma) = \#\mathrm{Ddes}\,(\sigma)$, $\mathrm{pasc}(\sigma) = \#\mathrm{Pasc}(\sigma)$, $\mathrm{apd}(\sigma) = \#\,\mathrm{Apd}(\sigma)$, $\mathrm{uu}(\sigma) = \#\,\mathrm{Uu}(\sigma)$ and $\mathrm{dd}(\sigma) = \#\,\mathrm{Dd}(\sigma)$.

Example 12. Let $\sigma = 77441223315665 \in \mathcal{Q}_7$. The corresponding SP-code is given by

$$C_7 = (0,0)(1,2)(2,3)(1,1)(1,3)(5,2)(4,1).$$

Table 1
The correspondences of statistics on Stirling permutations and SP-codes.

Statistics on Stirling permutation	Statistics on SP-code
Asc (ascent)	$[n] - \{a_i \mid (a_i, 1) \in C_n\}$
Plat (plateau)	$[n] - \{a_i \mid (a_i, 2) \in C_n\}$
Des (descent)	$[n] - \{a_i \mid (a_i, 3) \in C_n\}$
Lap (left ascent-plateau)	$[n] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 2) \in C_n\}$
Rpd (right plateau-descent)	$[n] - \{a_i \mid (a_i, 2) \text{ or } (a_i, 3) \in C_n\}$
Eud (exterior up-down-pair)	$[n] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 3) \in C_n\}$
Dasc (double ascent)	$\{a_i \mid (a_i, 1) \notin C_n \ \& \ (a_i, 2) \in C_n\}$
Dplat (descent-plateau)	$\{a_i \mid (a_i, 1) \in C_n \ \& \ (a_i, 2) \notin C_n\}$
Ddes (double descent)	$\{a_i \mid (a_i, 2) \in C_n \ \& \ (a_i, 3) \notin C_n\}$
Pasc (plateau-ascent)	$\{a_i \mid (a_i, 2) \notin C_n \ \& \ (a_i, 3) \in C_n\}$
Apd (ascent-plateau-descent)	$\{a_i \mid (a_i, 1) \notin C_n \ \& \ (a_i, 2) \notin C_n \ \& \ (a_i, 3) \notin C_n\}$
Uu (up-up-pair)	$\{a_i \mid (a_i, 1) \notin C_n \ \& \ (a_i, 3) \in C_n\}$
Dd (down-down-pair)	$\{a_i \mid (a_i, 1) \in C_n \ \& \ (a_i, 3) \notin C_n\}$

Then we have

$$\begin{aligned}
\mathrm{Asc}(\sigma) &= [7] - \{a_i \mid (a_i, 1) \in C_7\} = \{2, 3, 5, 6, 7\}, \\
\mathrm{Plat}(\sigma) &= [7] - \{a_i \mid (a_i, 2) \in C_7\} = \{2, 3, 4, 6, 7\}, \\
\mathrm{Des}(\sigma) &= [7] - \{a_i \mid (a_i, 3) \in C_7\} = \{3, 4, 5, 6, 7\}, \\
\mathrm{Lap}(\sigma) &= [7] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 2) \in C_7\} = \{2, 3, 6, 7\}, \\
\mathrm{Rpd}(\sigma) &= [7] - \{a_i \mid (a_i, 2) \text{ or } (a_i, 3) \in C_7\} = \{3, 4, 6, 7\}, \\
\mathrm{Eud}(\sigma) &= [7] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 3) \in C_7\} = \{3, 5, 6, 7\}, \\
\mathrm{Dasc}(\sigma) &= \{a_i \mid (a_i, 1) \notin C_7 \ \& \ (a_i, 2) \in C_7\} = \{5\}, \\
\mathrm{Dplat}(\sigma) &= \{a_i \mid (a_i, 1) \in C_7 \ \& \ (a_i, 2) \notin C_7\} = \{4\}, \\
\mathrm{Ddes}(\sigma) &= \{a_i \mid (a_i, 2) \in C_7 \ \& \ (a_i, 3) \notin C_7\} = \{5\}, \\
\mathrm{Pasc}(\sigma) &= \{a_i \mid (a_i, 2) \notin C_7 \ \& \ (a_i, 3) \in C_7\} = \{2\}, \\
\mathrm{Apd}(\sigma) &= \{a_i \mid (a_i, 1) \notin C_7 \ \& \ (a_i, 2) \notin C_7 \ \& \ (a_i, 3) \notin C_7\} = \{3, 6\}, \\
\mathrm{Uu}(\sigma) &= \{a_i \mid (a_i, 1) \notin C_7 \ \& \ (a_i, 3) \in C_7\} = \{2\}, \\
\mathrm{Dd}(\sigma) &= \{a_i \mid (a_i, 1) \in C_7 \ \& \ (a_i, 3) \notin C_7\} = \{4\}.
\end{aligned}$$

Combining the bijections ϕ (see subsection 1.2) and Γ (defined in the proof of Theorem 10), it is clear that the set-valued statistics on Stirling permutations listed in Table 1 correspond to the given set-valued statistics on SP-codes. We illustrate these correspondences in Example 12. By Table 1, a large number of equidistribution results can be deduced. The following two results generalize (2), which can be proved by switching some 2-tuples in the corresponding SP-codes.

Theorem 13. *The six bivariable set-valued statistics are all equidistributed on $\mathcal{Q}_n$:*

$$(\mathrm{Asc}, \mathrm{Dasc}), \quad (\mathrm{Plat}, \mathrm{Dplat}), \quad (\mathrm{Des}, \mathrm{Ddes}),$$

$$(\mathrm{Asc}, \mathrm{Uu}), \quad (\mathrm{Plat}, \mathrm{Pasc}), \quad (\mathrm{Des}, \mathrm{Dd}).$$

So we get the following four identities:

$$\begin{aligned}
\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)}y^{\mathrm{dasc}\,(\sigma)} &= \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{plat}\,(\sigma)}y^{\mathrm{dplat}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{des}\,(\sigma)}y^{\mathrm{ddes}\,(\sigma)},\\
\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)}y^{\mathrm{dasc}\,(\sigma)} &= \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{plat}\,(\sigma)}y^{\mathrm{pasc}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{des}\,(\sigma)}y^{\mathrm{ddes}\,(\sigma)},\\
\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)}y^{\mathrm{uu}(\sigma)} &= \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{plat}\,(\sigma)}y^{\mathrm{pasc}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{des}\,(\sigma)}y^{\mathrm{dd}(\sigma)},\\
\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)}y^{\mathrm{uu}(\sigma)} &= \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{plat}\,(\sigma)}y^{\mathrm{dplat}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{des}\,(\sigma)}y^{\mathrm{dd}(\sigma)}.
\end{aligned}$$

Proof. Consider Table 1. For $C_n \in \mathrm{CQ}_n$, if we switch the 2-tuples $(a_i, 1)$ and $(a_i, 2)$ for all i (if any), then we see that the following bivariable set-valued statistics are equidistributed on CQ_n:

$$\begin{aligned}
&([n] - \{a_i \mid (a_i,1) \in C_n\}, \{a_i \mid (a_i,1) \notin C_n \ \&\ (a_i,2) \in C_n\}),\\
&([n] - \{a_i \mid (a_i,2) \in C_n\}, \{a_i \mid (a_i,1) \in C_n \ \&\ (a_i,2) \notin C_n\}).
\end{aligned}$$

By Table 1, we obtain that $(\mathrm{Asc}, \mathrm{Dasc})$ and $(\mathrm{Plat}, \mathrm{Dplat})$ are equidistributed on $\mathcal{Q}_n$.

If we switch the 2-tuples $(a_i, 1)$ and $(a_i, 3)$ for all i (if any), then we find that the following bivariable set-valued statistics are equidistributed on CQ_n:

$$\begin{aligned}
&([n] - \{a_i \mid (a_i,1) \in C_n\}, \{a_i \mid (a_i,1) \notin C_n \ \&\ (a_i,2) \in C_n\}),\\
&([n] - \{a_i \mid (a_i,3) \in C_n\}, \{a_i \mid (a_i,2) \in C_n \ \&\ (a_i,3) \notin C_n\}).
\end{aligned}$$

By Table 1, we get that $(\mathrm{Asc}, \mathrm{Dasc})$ and $(\mathrm{Des}, \mathrm{Ddes})$ are equidistributed on $\mathcal{Q}_n$.

If we switch the 2-tuples $(a_i, 1)$ and $(a_i, 2)$ for all i (if any), then we see that the following bivariable set-valued statistics are equidistributed on CQ_n:

$$\begin{aligned}
&([n] - \{a_i \mid (a_i,1) \in C_n\}, \{a_i \mid (a_i,1) \notin C_n \ \&\ (a_i,3) \in C_n\}),\\
&([n] - \{a_i \mid (a_i,2) \in C_n\}, \{a_i \mid (a_i,2) \notin C_n \ \&\ (a_i,3) \in C_n\}).
\end{aligned}$$

By Table 1, we obtain that $(\mathrm{Asc}, \mathrm{Uu})$ and $(\mathrm{Plat}, \mathrm{Pasc})$ are equidistributed on $\mathcal{Q}_n$.

If we switch the 2-tuples $(a_i, 1)$ and $(a_i, 3)$ for all i (if any), then we get that the following bivariable set-valued statistics are equidistributed on CQ_n:

$$\begin{aligned}
&([n] - \{a_i \mid (a_i,1) \in C_n\}, \{a_i \mid (a_i,1) \notin C_n \ \&\ (a_i,3) \in C_n\}),\\
&([n] - \{a_i \mid (a_i,3) \in C_n\}, \{a_i \mid (a_i,1) \in C_n \ \&\ (a_i,3) \notin C_n\}).
\end{aligned}$$

By Table 1, we obtain that $(\mathrm{Asc}, \mathrm{Uu})$ and $(\mathrm{Des}, \mathrm{Dd})$ are equidistributed on $\mathcal{Q}_n$.

If we switch the 2-tuples $(a_i, 2)$ and $(a_i, 3)$ for all i (if any), then we find that the following bivariable set-valued statistics are equidistributed on CQ_n:

$$([n] - \{a_i \mid (a_i, 2) \in C_n\}, \{a_i \mid (a_i, 1) \in C_n \ \&\ (a_i, 2) \notin C_n\}),$$
$$([n] - \{a_i \mid (a_i, 3) \in C_n\}, \{a_i \mid (a_i, 1) \in C_n \ \&\ (a_i, 3) \notin C_n\}).$$

It follows from Table 1 that (Plat, Dplat) and (Des, Dd) are equidistributed on $\mathcal{Q}_n$.

In conclusion, the proof is completed by using transitivity. □

Theorem 14. *The six bivariable set-valued statistics are equidistributed on $\mathcal{Q}_n$:*

$$(\mathrm{Asc}, \mathrm{Lap}), \quad (\mathrm{Plat}, \mathrm{Lap}), \quad (\mathrm{Des}, \mathrm{Rpd}),$$
$$(\mathrm{Asc}, \mathrm{Eud}), \quad (\mathrm{Plat}, \mathrm{Rpd}), \quad (\mathrm{Des}, \mathrm{Eud}).$$

So we get the following six identities:

$$\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)} y^{\mathrm{lap}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{plat}\,(\sigma)} y^{\mathrm{lap}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{des}\,(\sigma)} y^{\mathrm{rpd}(\sigma)},$$
$$\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)} y^{\mathrm{lap}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{plat}\,(\sigma)} y^{\mathrm{rpd}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{des}\,(\sigma)} y^{\mathrm{rpd}(\sigma)},$$
$$\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)} y^{\mathrm{eud}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{plat}\,(\sigma)} y^{\mathrm{rpd}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{des}\,(\sigma)} y^{\mathrm{eud}(\sigma)},$$
$$\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)} y^{\mathrm{eud}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{plat}\,(\sigma)} y^{\mathrm{lap}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{des}\,(\sigma)} y^{\mathrm{eud}(\sigma)},$$
$$\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)} y^{\mathrm{lap}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{plat}\,(\sigma)} y^{\mathrm{rpd}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{des}\,(\sigma)} y^{\mathrm{eud}(\sigma)},$$
$$\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)} y^{\mathrm{eud}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{plat}\,(\sigma)} y^{\mathrm{lap}(\sigma)} = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{des}\,(\sigma)} y^{\mathrm{rpd}(\sigma)},$$

where the last two identities generalize (2) and (7) simultaneously.

Proof. We only prove that (Asc, Lap) and (Plat, Lap) are equidistributed on $\mathcal{Q}_n$. The other pairs can be proved in the same way. For $C_n \in \mathrm{CQ}_n$, if we switch the 2-tuples $(a_i, 1)$ and $(a_i, 2)$ for all i (if any), then the following bivariable set-valued statistics are equidistributed on CQ_n:

$$([n] - \{a_i \mid (a_i, 1) \in C_n\}, [n] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 2) \in C_n\}),$$
$$([n] - \{a_i \mid (a_i, 2) \in C_n\}, [n] - \{a_i \mid (a_i, 1) \text{ or } (a_i, 2) \in C_n\}).$$

By Table 1, we get that (Asc, Lap) and (Plat, Lap) are equidistributed on $\mathcal{Q}_n$. □

We say that a joint distribution of (set-valued) statistics or a multivariate polynomial is *symmetric* if it is invariant under any permutation of its indeterminates. By Table 1, we can now present the following two results, and we omit the proofs for simplicity.

Theorem 15. *The six set-valued statistics are all equidistributed on $\mathcal{Q}_n$:*

$$\mathrm{Dasc}, \mathrm{Dplat}, \mathrm{Ddes}, \mathrm{Pasc}, \mathrm{Uu}, \mathrm{Dd}\,.$$

Moreover, if we select any two set-valued statistics from these six set-valued statistics, then the selected two set-valued statistics are symmetric on $\mathcal{Q}_n$.

Theorem 16. *The following triple set-valued statistics are all symmetric on $\mathcal{Q}_n$:*

$$(\mathrm{Asc}(\sigma), \mathrm{Plat}(\sigma), \mathrm{Des}(\sigma)),\ \ (\mathrm{Lap}(\sigma), \mathrm{Rpd}(\sigma), \mathrm{Eud}(\sigma)),$$
$$(\mathrm{Dasc}(\sigma), \mathrm{Pasc}(\sigma), \mathrm{Dd}(\sigma)),\ \ (\mathrm{Ddes}(\sigma), \mathrm{Dplat}(\sigma), \mathrm{Uu}(\sigma)).$$

Here we give an example to illustrate the symmetry of the joint distribution of the set-valued statistics Ddes and Pasc.

Example 17. Let σ and C_7 be the given in Example 12. Then $\mathrm{Ddes}(\sigma) = \{5\}$ and $\mathrm{Pasc}(\sigma) = \{2\}$. Let Φ be the bijection on CQ_7 that is defined by

$$(a_i, 2) \leftrightarrow (a_i, 3),\ \text{ where } 1 \leqslant i \leqslant 6.$$

In other words, we just switch the 2-tuples $(a_i, 2)$ and $(a_i, 3)$ for all i (if any). Thus

$$\Phi\left((0,0)(1,2)(2,3)(1,1)(1,3)(5,2)(4,1)\right) = (0,0)(1,3)(2,2)(1,1)(1,2)(5,3)(4,1).$$

It is easy to verify that $\phi^{-1}\left(\Phi(C_7)\right) = 774415\mathbf{5}661233\mathbf{2}$. Therefore, we have

$$\mathrm{Ddes}\left(\phi^{-1}\left(\Phi(C_7)\right)\right) = \{2\},\ \ \mathrm{Pasc}\left(\phi^{-1}\left(\Phi(C_7)\right)\right) = \{5\}.$$

Corollary 18. *The following two polynomials are both symmetric in their variables:*

$$C_n(x,y,z) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)} y^{\mathrm{plat}\,(\sigma)} z^{\mathrm{des}\,(\sigma)},$$
$$N_n(x,y,z) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{lap}\,\sigma)} y^{\mathrm{rpd}(\sigma)} z^{\mathrm{eud}(\sigma)}.$$

As discussed in the introduction, the symmetry of $C_n(x,y,z)$ has been extensively studied, see [8,16] for instance. In Section 5, we shall show the e-positivity of $N_n(x,y,z)$.

4. Bijections among SP-codes, trapezoidal words and perfect matchings

Following Riordan [30], we say that a word $t = t_1t_2\cdots t_n$ is a *Riordan trapezoidal word* if the element t_i takes the values $1, 2, \ldots, 2i-1$ for all $1 \leqslant i \leqslant n$. Let RT_n be the set of Riordan trapezoidal words of length n. In particular, Besides (6), Dumont [10] gave interpretations of $C_n(x,y,z)$ in terms of Dumont trapezoidal words as well as perfect matchings. The Dumont trapezoidal word [10] is a variant of the Riordan trapezoidal word. A word $w = w_1w_2\cdots w_n$ is called a *Dumont trapezoidal word* of length n if $0 \leqslant |w_i| < i$ for all $1 \leqslant i \leqslant n$, where w_i are all integers. Let DT_n denote the set of Dumont trapezoidal words of length n. As usual, we write $\overline{i} = -i$. For example,

$$\mathrm{RT}_1 = \{1\},\ \mathrm{RT}_2 = \{11, 12, 13\},\ \mathrm{DT}_1 = \{0\},\ \mathrm{DT}_2 = \{00, 01, 0\overline{1}\}.$$

Given $w \in \mathrm{DT}_n$. Let $\mathrm{dist}(w)$ be the number of distinct elements in w, and we define

$$\mathrm{nneg}(\sigma) = n - \{w_i \mid w_i < 0\},\ \mathrm{npos}(\sigma) = n - \{w_i \mid w_i > 0\}.$$

Dumont [10, Section 2.3] found that

$$C_n(x,y,z) = \sum_{w\in\in\mathrm{DT}_n} x^{\mathrm{dist}(w)} y^{\mathrm{nneg}(\sigma)} z^{\mathrm{npos}(\sigma)}.$$

A *perfect matching* of $[2n]$ is a set partition of $[2n]$ with blocks (disjoint nonempty subsets) of size exactly 2. Let $\mathcal{M}_{2n}$ be the set of perfect matchings of $[2n]$, and let $\mathrm{M} \in \mathcal{M}_{2n}$. The *standard form* of M is a list of blocks $\{(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)\}$ such that $i_r < j_r$ for all $1 \leqslant r \leqslant n$ and $1 = i_1 < i_2 < \cdots < i_n$. In this paper, we always write M in standard form. It is well known that M can be regarded as a fixed-point-free involution on $[2n]$. In particular,

$$\mathcal{M}_2 = \{(1,2)\},\ \mathcal{M}_4 = \{(1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}.$$

As a continuation of Theorem 10, it is natural to explore the bijections among SP-codes, trapezoidal words and perfect matchings.

Theorem 19. *For $n \geqslant 1$, we have*

$$\mathrm{CQ}_n \cong \mathrm{DT}_n \cong \mathrm{RT}_n \cong \mathcal{M}_{2n}. \tag{11}$$

Proof. (i) Let $\varphi_1 : \mathrm{DT}_n \to \mathrm{RT}_n$ be the bijection defined by

$$\varphi_1(w_i) = \begin{cases} 1, & \text{if } w_i = 0; \\ 2k, & \text{if } w_i = k > 0; \\ 2k+1, & \text{if } w_i = \overline{k} < 0, \end{cases} \tag{12}$$

which yields that $\mathrm{DT}_n \cong \mathrm{RT}_n$.

(ii) We now start to construct a bijection, denoted by φ_2, from RT_n to $\mathcal{M}_{2n}$. Recall that $\mathrm{RT}_1 = \{1\}$. Set $\varphi_2(\mathbf{1}) = (\mathbf{1}, 2)$. When $n = 2$, $\mathrm{RT}_2 = \{11, 12, 13\}$, and we set

$$\varphi_2(1\mathbf{1}) = (\mathbf{1}, 4)(2, 3),\ \varphi_2(1\mathbf{2}) = (\mathbf{2}, 4)(1, 3),\ \varphi_2(1\mathbf{3}) = (\mathbf{3}, 4)(1, 2).$$

We proceed by induction. Let $n = m$. Suppose that φ_2 is a bijection from RT_m to $\mathcal{M}_{2m}$. Given $\mathrm{M} = (i_1, j_1)(i_2, j_2)\cdots(i_m, j_m) \in \mathcal{M}_{2m}$. Suppose that $\varphi_2(t) = \mathrm{M}$, where $t = t_1t_2\cdots t_m \in \mathrm{RT}_m$. For $1 \leqslant i \leqslant 2m + 1$, the map φ_2 is defined as follows:

- $\varphi_2(t_1t_2\cdots t_m i) = (i, 2m+2)(i'_1, j'_1)(i'_2, j'_2)\cdots(i'_m, j'_m)$, where $(i'_1, j'_1)(i'_2, j'_2)\cdots(i'_m, j'_m)$ is a perfect matching of $[2m + 2] - \{i, 2m + 2\}$ such that the elements in $(i'_1, j'_1)\cdots(i'_m, j'_m)$ keep the same order relationships they had in $(i_1, j_1)\cdots(i_m, j_m)$.

Clearly, φ_2 is the desired bijection.

(iii) Now we start to construct a bijection, denoted by φ_3, from DT_n to CQ_n. When $n = 1$, we set $\varphi_3(0) = (0, 0)$. When $n \leqslant m$, suppose φ_3 is a bijection from DT_n to CQ_n. Consider the case $n = m + 1$. Let $w = w_1w_2\cdots w_{m+1} \in \mathrm{DT}_{m+1}$. Then $w' = w_1w_2\cdots w_m \in \mathrm{DT}_m$ and $\varphi_3(w') = ((0, 0), (a_1, b_1), (a_2, b_2)\ldots, (a_{m-1}, b_{m-1})) \in \mathrm{CQ}_m$. We distinguish three cases:

(c_1) $w_{m+1} = k$ and $k \in \{w_1, w_2, \ldots, w_m\}$ if and only if

$$(a_m, b_m) = (j, 1),\ \text{where } j = \max\{i \mid w_i = k,\ 1 \leqslant i \leqslant m\};$$

(c_2) $w_{m+1} = -j$ and $-j \notin \{w_1, w_2, \ldots, w_m\}$ if and only if $(a_m, b_m) = (j, 2)$, where $1 \leqslant j \leqslant m$;

(c_3) $w_{m+1} = j$ and $j \notin \{w_1, w_2, \ldots, w_m\}$ if and only if $(a_m, b_m) = (j, 3)$, where $1 \leqslant j \leqslant m$.

It is routine to check that φ_3 is the desired bijection. When $n = 2, 3$, we have

$$\begin{gathered}
\varphi_3(00) = (0,0)(1,1),\ \varphi_3(0\overline{1}) = (0,0)(1,2),\ \varphi_3(01) = (0,0)(1,3); \\
\varphi_3(000) = (0,0)(1,1)(2,1),\ \varphi_3(00\overline{1}) = (0,0)(1,1)(1,2),\ \varphi_3(001) = (0,0)(1,1)(1,3), \\
\varphi_3(00\overline{2}) = (0,0)(1,1)(2,2),\ \varphi_3(002) = (0,0)(1,1)(2,3),\ \varphi_3(0\overline{1}0) = (0,0)(1,2)(1,1), \\
\varphi_3(0\overline{1}1) = (0,0)(1,2)(1,3),\ \varphi_3(0\overline{1}\,\overline{1}) = (0,0)(1,2)(2,1),\ \varphi_3(0\overline{1}\,\overline{2}) = (0,0)(1,2)(2,2), \\
\varphi_3(0\overline{1}2) = (0,0)(1,2)(2,3),\ \varphi_3(010) = (0,0)(1,3)(1,1),\ \varphi_3(01\overline{1}) = (0,0)(1,3)(1,2), \\
\varphi_3(011) = (0,0)(1,3)(2,1),\ \varphi_3(01\overline{2}) = (0,0)(1,3)(2,2),\ \varphi_3(012) = (0,0)(1,3)(2,3).
\end{gathered}$$

This completes the proof. □

As an illustration of φ_3, we give an example.

Example 20. Given $w = 0\text{-}0\text{-}0\text{-}\overline{1}\text{-}1\text{-}5\text{-}1 \in \mathrm{DT}_7$. We give the procedure of creating $\varphi_3(w)$.

$$\begin{gathered}
\mathbf{0} \Leftrightarrow (0,0), \\
0\text{-}\mathbf{0} \Leftrightarrow (0,0)(\mathbf{1},\mathbf{1}), \\
0\text{-}0\text{-}\mathbf{0} \Leftrightarrow (0,0)(1,1)(\mathbf{2},\mathbf{1}), \\
0\text{-}0\text{-}0\text{-}\overline{\mathbf{1}} \Leftrightarrow (0,0)(1,1)(2,1)(\mathbf{1},\mathbf{2}), \\
0\text{-}0\text{-}0\text{-}\overline{1}\text{-}\mathbf{1} \Leftrightarrow (0,0)(1,1)(2,1)(1,2)(\mathbf{1},\mathbf{3}), \\
0\text{-}0\text{-}0\text{-}\overline{1}\text{-}1\text{-}5 \Leftrightarrow (0,0)(1,1)(2,1)(1,2)(1,3)(\mathbf{5},\mathbf{3}), \\
0\text{-}0\text{-}0\text{-}\overline{1}\text{-}1\text{-}5\text{-}\mathbf{1} \Leftrightarrow (0,0)(1,1)(2,1)(1,2)(1,3)(5,3)(\mathbf{5},\mathbf{1}).
\end{gathered}$$

Thus $\varphi_3(w) = (0,0)(1,1)(2,1)(1,2)(1,3)(5,3)(5,1) \in \mathrm{CQ}_7$. Conversely, $\varphi_3^{-1}(\varphi_3(w)) = w$.

5. The e-positivity of $N_n(x,y,z)$

5.1. Preliminary

Let $\mathrm{X}_n = \{x_1, x_2, \ldots, x_n\}$ be a set of commuting variables. Define

$$S_n(x) = \prod_{i=1}^{n}(x - x_i) = \sum_{k=0}^{n}(-1)^k e_k x^{n-k}.$$

Then the *k-th elementary symmetric function* associated with X_n is defined by

$$e_k = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

In particular, $e_0 = 1$, $e_1 = \sum_{i=1}^{n} x_i$, $e_n = x_1 x_2 \cdots x_n$. A function $f(x_1, x_2, \ldots) \in \mathbb{R}[x_1, x_2, \ldots]$ is said to be *symmetric* if it is invariant under any permutation of its indeterminates. We say that a symmetric function is *e-positive* if it can be written as a nonnegative linear combination of elementary symmetric functions.

For an alphabet A, let $\mathbb{Q}[[A]]$ be the rational commutative ring of formal power series in monomials formed from letters in A. Following Chen [6], a *context-free grammar* over A is a function $G : A \to \mathbb{Q}[[A]]$ that replaces each letter in A by a formal function over A. The formal derivative D_G with respect to G satisfies the derivation rules:

$$D_G(u+v) = D_G(u) + D_G(v),\ D_G(uv) = D_G(u)v + uD_G(v).$$

So the Leibniz rule holds:

$$D_G^n(uv)=\sum_{k=0}^{n}\binom{n}{k}D_G^k(u)D_G^{n-k}(v).$$

See [11,26] for some examples of context-free grammars.

Recently, two methods are developed in the combinatorial theory of context-free grammars, i.e., grammatical labeling and the change of grammars. A *grammatical labeling* is an assignment of the underlying elements of a combinatorial structure with variables, which is consistent with the substitution rules of a grammar (see [7] for details). The *change of grammars* is a substitution method in which the original grammars are replaced with functions of other grammars. In particular, the following change of grammars can be used to study the γ-positivity and partial γ-positivity of enumerative polynomials (see [8,26,27] for details):

$$\begin{cases} u=xy, \\ v=x+y. \end{cases}$$

Let G be the following grammar

$$G=\{x\rightarrow xyz, y\rightarrow xyz, z\rightarrow xyz\}. \tag{13}$$

Dumont [10], Haglund-Visontai [16] and Chen-Hao-Yang [9] (in equivalent forms) showed that

$$D_G^n(x)=C_n(x,y,z).$$

Very recently, Chen-Fu [8] introduced a new change of grammars:

$$\begin{cases} u=x+y+z, \\ v=xy+yz+zx, \\ w=xyz. \end{cases} \tag{14}$$

Combining (13) and (14), one can easily verify that $D_G(u)=3w$, $D_G(v)=2uw$, $D_G(w)=vw$. So we get a new grammar

$$H=\{u\rightarrow 3w, v\rightarrow 2uw,\ w\rightarrow vw\}. \tag{15}$$

For any $n\geqslant 1$, Chen-Fu [8] discovered that

$$C_n(x,y,z)=D_G^n(x)=D_H^{n-1}(w)=\sum_{i+2j+3k=2n+1}\gamma_{n,i,j,k}u^iv^jw^k, \tag{16}$$

where the coefficient $\gamma_{n,i,j,k}$ is defined in (5). We can now present the main result of this section.

Theorem 21. *For* $n \geqslant 1$, *let*

$$N_n(x,y,z) = \sum_{\sigma \in \mathcal{Q}_n} x^{\operatorname{lap}(\sigma)} y^{\operatorname{eud}(\sigma)} z^{\operatorname{rpd}(\sigma)}.$$

Then we have

$$N_n(x,y,z) = \sum_{i+2j+3k=2n+1} 3^i \gamma_{n,i,j,k} (x+y+z)^j (xyz)^k, \tag{17}$$

where the coefficient $\gamma_{n,i,j,k}$ *is the same as in* (16), *i.e.,* $\gamma_{n,i,j,k}$ *equals the number of 0-1-2-3 increasing plane trees on* $[n]$ *with* k *leaves,* j *vertices with degree one and* i *degree two vertices.*

Throughout this section, we always set $w_1 = x+y+z,\ w_2 = xy+yz+zx,\ w_3 = xyz$. Below are the polynomials $N_n(x,y,z)$ for $n \leqslant 6$:

$$\begin{aligned}
&N_1(x,y,z) = w_3,\ N_2(x,y,z) = w_1w_3,\ N_3(x,y,z) = w_1^2w_3 + 6w_3^2,\\
&N_4(x,y,z) = w_1^3w_3 + 24w_1w_3^2 + 6w_3^3,\\
&N_5(x,y,z) = w_1^4w_3 + 66w_1^2w_3^2 + 42w_1w_3^3 + 144w_3^3,\\
&N_6(x,y,z) = w_1^5w_3 + 156w_1^3w_3^2 + 192w_1^2w_3^3 + 1224w_1w_3^3 + 540w_3^4.
\end{aligned}$$

Example 22. For the elements in $\mathcal{Q}_2$, we have

$$\begin{aligned}
&\operatorname{lap}(1122) = \operatorname{lap}(011220) = 2,\ \operatorname{eud}(1122) = \operatorname{eud}(011220) = 1,\\
&\operatorname{rpd}(1122) = \operatorname{rpd}(011220) = 1,\ \operatorname{lap}(1221) = \operatorname{lap}(012210) = 1,\\
&\operatorname{eud}(1221) = \operatorname{eud}(012210) = 2,\ \operatorname{rpd}(1221) = \operatorname{rpd}(012210) = 1,\\
&\operatorname{lap}(2211) = \operatorname{lap}(022110) = 1,\ \operatorname{eud}(2211) = \operatorname{eud}(022110) = 1,\\
&\operatorname{rpd}(2211) = \operatorname{rpd}(022110) = 2.
\end{aligned}$$

Thus $N_2(x,y,z) = xyz(x+y+z)$. See Fig. 4 for an illustration, where the weights are explained in (20).

5.2. Proof of Theorem 21

As discussed in Section 3, we shall use simplified ternary increasing trees. See Fig. 4 for an illustration, where the left figure represents the three different figures in the right. The weights $E_1(\sigma), E_2(C_n)$ of $\sigma \in \mathcal{Q}_n$ and $C_n \in \mathrm{CQ}_n$ are respectively defined as follows:

$$E_1(\sigma) = x^{\operatorname{lap}(\sigma)} y^{\operatorname{eud}(\sigma)} z^{\operatorname{rpd}(\sigma)},$$

$$E_2(C_n) = x^{n-\#\{a_i \mid (a_i,1) \text{ or } (a_i,2) \in C_n\}} y^{n-\#\{a_i \mid (a_i,1) \text{ or } (a_i,3) \in C_n\}} z^{n-\#\{a_i \mid (a_i,2) \text{ or } (a_i,3) \in C_n\}}. \tag{18}$$

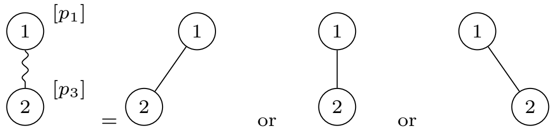

Fig. 4. $N_2(x, y, z) = (x + y + z)xyz = p_1p_3$.

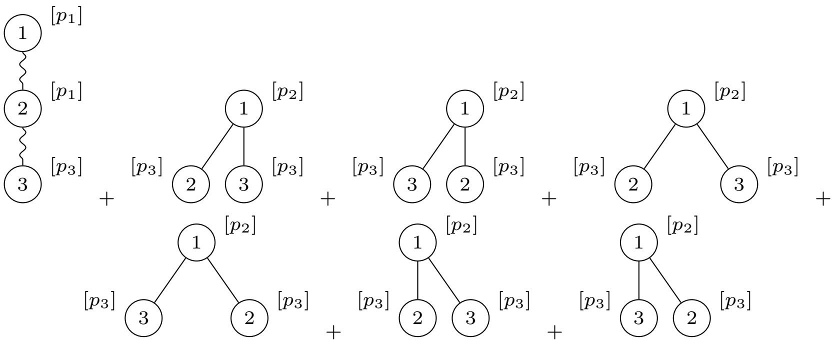

Fig. 5. $N_3(x, y, z) = (x + y + z)^2xyz + 6(xyz)^2 = p_1^2p_3 + 6p_3^2$.

Assume that C_n is the corresponding SP-code of σ. It follows from Table 1 that $E_1(\sigma) = E_2(C_n)$. The SP-code $(0, 0)$ corresponds to the Stirling permutation 11. Clearly, $E_1(11) = E_2((0, 0)) = xyz = w_3$. When $n = 2$, the weights of elements in $\mathcal{Q}_2$ and CQ_n are respectively given as follows:

$$\underbrace{2211 \leftrightarrow (0,0)(1,1)}_{xyz^2=w_3z},\ \underbrace{1221 \leftrightarrow (0,0)(1,2)}_{xy^2z=w_3y},\ \underbrace{1122 \leftrightarrow (0,0)(1,3)}_{x^2yz=w_3x},$$

and the sum of weights is given by $w_3(x + y + z) = w_3w_1$.

Given $C_n = (0, 0)(a_1, b_1)(a_2, b_2) \cdots (a_{n-1}, b_{n-1}) \in \mathrm{CQ}_n$. Consider the elements in CQ_{n+1} generated from C_n by appending the 2-tuples (a_n, b_n), where $1 \leqslant a_n \leqslant n$ and $1 \leqslant b_n \leqslant 3$. Let T be the corresponding ternary increasing tree of C_n. We can add $n+1$ to T as a child of a vertex, which is not of degree three. Let T' be the resulting ternary increasing tree. We first give a labeling of T as follows. Label a leaf of T by p_3, a degree one vertex by p_1, a degree two vertex by p_2 and a degree three vertex by 1.

The 2-tuples (a_n, b_n) can be divided into three classes:

- if $a_n \neq a_i$ for all $1 \leqslant i \leqslant n-1$, then we must add $n+1$ to a leaf of T. This operation corresponds to the change of weights

$$E_2(C_n) \to E_2(C_{n+1}) = E_2(C_n)(x + y + z), \tag{19}$$

which yields the substitution $p_3 \to p_1p_3$, see Fig. 4 and the first case in Fig. 5 for illustrations. Thus the contribution of any leaf to the weight is xyz and that of a degree one vertex is $x+y+z$ (which represents that this vertex may have a left child, a middle child or a right child). When we compute the corresponding enumerative polynomials of Stirling permutations, it follows from (19) that we need to set

$$p_1 = x + y + z,\ p_3 = xyz; \tag{20}$$

- if there is exactly one 2-tuple (a_i, b_i) in C_n such that $a_n = a_i$, then we must add $n+1$ to T as a child of the node a_i. Note that the node a_i already has the child $i+1$, and $n+1$ becomes the second child of a_i. There are six cases to add $n+1$. As illustrations, the last six cases in Fig. 5 are the total possibilities when we add 3 to the simplified ternary increasing trees in Fig. 4 as the second child of the node 1. This operation corresponds to the substitution $p_1 \to 6p_2p_3$. From (18), we see that each degree two vertex makes no contribution to the weight. Thus we need to set $p_2 = 1$ when we compute the corresponding enumerative polynomial of the joint distribution of $(\mathrm{lap}, \mathrm{eud}, \mathrm{rpd})$;
- if there are exactly two 2-tuples (a_i, b_i) and (a_j, b_j) in C_n such that $a_n = a_i = a_j$ and $i < j$, then we must add $n+1$ to T as the third child of a_i, and $n+1$ becomes a leaf with label p_3. This operation corresponds to the substitution $p_2 \to p_3$. From (18), we see that each degree three vertex makes no contribution to the weight, and so we label each degree three vertex by 1.

The aforementioned three cases exhaust all the possibilities to construct SP-codes of length $n+1$ from a SP-code of length n by appending 2-tuples (a_n, b_n). In conclusion, each case corresponds to an application of the substitution rules defined by the following grammar:

$$I = \{p_3 \to p_1p_3,\ p_1 \to 6p_2p_3,\ p_2 \to p_3\}. \tag{21}$$

We can now conclude the following lemma.

Lemma 23. *Let I be the context-free grammar given by (21). For any $n \geqslant 1$, we have*

$$D_I^{n-1}(p_3)\ |_{p_1=x+y+z,p_2=1,p_3=xyz}= N_n(x, y, z).$$

In particular, $D_I(p_3) = p_1p_3$, $D_I^2(p_3) = p_1^2p_3+6p_2p_3^2$ and $D_I^3(p_3) = p_1^3p_3+24p_1p_2p_3^2+6p_3^3$.

Proof of Theorem 21. For the grammar H defined by (15), set $w = p_3$, $v = p_1$ and $u = 3p_2$, we get $D_H(p_3) = p_1p_3$, $D_H(p_1) = 6p_2p_3$, $D_H(p_2) = p_3$, which yields the grammar I. It follows from (16) that

$$D_I^{n-1}(p_3) = D_H^{n-1}(w)\ |_{w=p_3,v=p_1,u=3p_2}= \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k} 3^i p_2^i p_1^j p_3^k.$$

By (20) and Lemma 23, we obtain

$$N_n(x,y,z) = \sum_{i+2j+3k=2n+1} 3^i \gamma_{n,i,j,k}(x+y+z)^j (xyz)^k.$$

This completes the proof of Theorem 21. □

6. The e-positivity of the multivariate k-th order Eulerian polynomials

6.1. Preliminary

A bivariate version of the Eulerian polynomial over the symmetric group is given as follows:

$$A_n(x,y) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{asc}\,(\pi)} y^{\mathrm{des}\,(\pi)}.$$

Clearly, $A_n(x,1) = A_n(1,x) = A_n(x)$. Carlitz and Scoville [5] showed that

$$A_{n+1}(x,y) = xy\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) A_n(x,y) \text{ with } A_1(x,y) = xy.$$

Foata and Schützenberger [14] found that $A_n(x,y)$ has the gamma-expansion

$$A_n(x,y) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \gamma(n,k)(xy)^k (x+y)^{n+1-2k},$$

where $\gamma(n,k)$ counts permutations in $\mathfrak{S}_n$ with k descents, but with no double descents.

In this section, we always let k be a given positive integer. A *k-Stirling permutation* of order n is a multiset permutation of $\{1^k, 2^k, \ldots, n^k\}$ with the property that all elements between two occurrences of i are at least i, where $i \in [n]$, see [21,22,33] for the recent study on k-Stirling permutations and their variants. Let $\mathcal{Q}_n(k)$ be the set of k-Stirling permutations of order n. It is clear that $\mathcal{Q}_n(1) = \mathfrak{S}_n,\ \mathcal{Q}_n(2) = \mathcal{Q}_n$.

Let $\sigma \in \mathcal{Q}_n(k)$. The ascents, descents and plateaux of σ are defined as before, where we always set $\sigma_0 = \sigma_{kn+1} = 0$. More precisely, an index i is called an ascent (resp. descent, plateau) of σ if $\sigma_i < \sigma_{i+1}$ (resp. $\sigma_i > \sigma_{i+1}$, $\sigma_i = \sigma_{i+1}$). Clearly,

$$\mathrm{asc}\,(\sigma) + \mathrm{des}\,(\sigma) + \mathrm{plat}\,(\sigma) = kn + 1.$$

As a natural refinement of ascents, descents and plateaux, Janson-Kuba-Panholzer [19] introduced the following definition, and related the distribution of j-ascents, j-descents and j-plateaux in k-Stirling permutations with certain parameters in $(k+1)$-ary increasing trees.

Definition 24 *([19]).* An index i is called a j-plateau (resp. j-descent, j-ascent) if i is a plateau (resp. descent, ascent) and there are exactly $j-1$ indices $\ell < i$ such that $a_\ell = a_i$.

Let $\operatorname{plat}_j(\sigma)$ be the number of j-plateaux of σ. For $\sigma \in \mathcal{Q}_n(k)$, it is clear that $\operatorname{plat}_j(\sigma) \leqslant k-1$.

Example 25. Consider the 4-Stirling permutation $\sigma = 111223333221$. The set of 1-plateaux is given by $\{1,4,6\}$, the set of 2-plateaux is given by $\{2,7\}$, and the set of 3-plateaux is given by $\{8,10\}$. Thus $\operatorname{plat}_1(\sigma) = 3$ and $\operatorname{plat}_2(\sigma) = \operatorname{plat}_3(\sigma) = 2$.

6.2. Main results

The *multivariate k-th order Eulerian polynomials* $C_n(x_1,\ldots,x_{k+1})$ are defined by

$$C_n(x_1,x_2,\ldots,x_{k+1}) = \sum_{\sigma\in\mathcal{Q}_n(k)} x_1^{\operatorname{plat}_1(\sigma)} x_2^{\operatorname{plat}_2(\sigma)} \cdots x_{k-1}^{\operatorname{plat}_{k-1}(\sigma)} x_k^{\operatorname{des}(\sigma)} x_{k+1}^{\operatorname{asc}(\sigma)}.$$

Some known enumerative polynomials are the special cases of this new polynomial. In particular, when $x_1 = z,\ x_2 = \cdots = x_{k-1} = 0,\ x_k = y$ and $x_{k+1} = x$, the polynomial $C_n(x_1,x_2,\ldots,x_{k+1})$ reduces to $C_n(x,y,z)$; when $x_1 = x_2 = \cdots = x_{k-1} = 0,\ x_k = 1$ and $x_{k+1} = x$, the polynomial $C_n(x_1,x_2,\ldots,x_{k+1})$ reduces to the Eulerian polynomial $A_n(x)$.

In the rest of this section, we always set $\mathrm{X}_{k+1} = \{x_1,x_2,\ldots,x_{k+1}\}$, and let e_i be the i-th elementary symmetric function associated with X_{k+1}. In particular,

$$e_0 = 1,\ e_1 = x_1 + x_2 + \cdots + x_{k+1},\ e_k = \sum_{i=1}^{k+1} \frac{e_{k+1}}{x_i},\ e_{k+1} = x_1x_2\cdots x_{k+1}.$$

The following lemma is fundamental.

Lemma 26. *Let $G_1 = \{x_1 \to e_{k+1},\ x_2 \to e_{k+1},\ldots,\ x_{k+1} \to e_{k+1}\}$ be a grammar, where $e_{k+1} = x_1x_2\cdots x_{k+1}$. For $n \geqslant 1$, one has $D_{G_1}^n(x_1) = C_n(x_1,x_2,\ldots,x_{k+1})$.*

Proof. We start to show that the grammar G_1 can be used to generate k-Stirling permutations. We first introduce a grammatical labeling of $\sigma \in \mathcal{Q}_n(k)$ as follows:

(L_1) If i is an ascent, then put a superscript label x_{k+1} right after σ_i;
(L_2) If i is a descent, then put a superscript label x_k right after σ_i;
(L_3) If i is a j-plateau, then put a superscript label x_j right after σ_i.

The weight of σ is defined as the product of the labels, that is

$$w(\sigma) = x_1^{\operatorname{plat}_1(\sigma)} x_2^{\operatorname{plat}_2(\sigma)} \cdots x_{k-1}^{\operatorname{plat}_{k-1}(\sigma)} x_k^{\operatorname{des}(\sigma)} x_{k+1}^{\operatorname{asc}(\sigma)}.$$

Recall that we always set $\sigma_0 = \sigma_{kn+1} = 0$. Thus the index 0 is always an ascent and the index kn is always a descent. Thus $\mathcal{Q}_1(k) = \{{}^{x_{k+1}}1^{x_1}1^{x_2}1^{x_3}\cdots 1^{x_{k-1}}1^{x_k}\}$. There are $k+1$ elements in $\mathcal{Q}_2(k)$ and they can be labeled as follows, respectively:

$$\begin{gathered} {}^{x_{k+1}}1^{x_1}1^{x_2}\cdots 1^{x_{k-1}}1^{x_{k+1}}2^{x_1}2^{x_2}\cdots 2^{x_{k-1}}2^{x_k}, \\ {}^{x_{k+1}}1^{x_1}1^{x_2}\cdots 1^{x_{k-2}}1^{x_{k+1}}2^{x_1}2^{x_2}\cdots 2^{x_{k-1}}2^{x_k}1^{x_k}, \quad \cdots \\ {}^{x_{k+1}}2^{x_1}2^{x_2}\cdots 2^{x_{k-1}}2^{x_k}1^{x_1}1^{x_2}\cdots 1^{x_{k-1}}1^{x_k}. \end{gathered}$$

Note that $D_{G_1}(x_1) = e_{k+1}$ and $D^2_{G_1}(x_1) = D_{G_1}(e_{k+1}) = e_ke_{k+1}$. Then the weight of the element in $\mathcal{Q}_1(k)$ is given by $D_{G_1}(x_1)$, and the sum of weights of the elements in $\mathcal{Q}_2(k)$ is given by $D^2_{G_1}(x)$. Hence the result holds for $n = 1, 2$. We proceed by induction on n. Suppose we get all labeled elements in $\mathcal{Q}_{n-1}(k)$, where $n \geqslant 3$. Let σ' be obtained from $\sigma \in \mathcal{Q}_{n-1}(k)$ by inserting the string $nn\cdots n$ with length k. The changes of the labeling can be illustrated as follows:

$$\begin{gathered} \cdots\sigma_i^{x_j}\sigma_{i+1}\cdots \mapsto \cdots\sigma_i^{x_{k+1}}n^{x_1}n^{x_2}\cdots n^{x_k}\sigma_{i+1}\cdots \text{ for any } 1 \leqslant j \leqslant k-1; \\ \sigma^{x_k} \mapsto \sigma^{x_{k+1}}n^{x_1}n^{x_2}\cdots n^{x_k}; \qquad {}^{x_{k+1}}\sigma \mapsto {}^{x_{k+1}}\, n^{x_1}n^{x_2}\cdots n^{x_k}\sigma. \end{gathered}$$

In the second case, we put the string $nn\cdots n$ at the end of σ. In the third case, we put the string $nn\cdots n$ at the front of σ. In each case, the insertion of the string $nn\cdots n$ corresponds to one substitution rule in G_1. Thus the action of D_{G_1} on the set of weights of all elements in $\mathcal{Q}_{n-1}(k)$ gives the set of weights of all elements in $\mathcal{Q}_n(k)$. In conclusion, we get the grammatical description of $C_n(x_1, x_2, \ldots, x_{k+1})$. $\square$

It should be noted that in [19] no explicit connection to the k-th order Eulerian polynomials has been established. By combining an urn model for the exterior leaves of $(k+1)$-ary increasing trees and a bijection between $(k+1)$-ary increasing trees and k-Stirling permutations, Janson-Kuba-Panholzer [19, Theorem 2, Theorem 8] found that the variables in $C_n(x_1, x_2, \ldots, x_{k+1})$ are exchangeable. We can now present the main result of this section.

Theorem 27. *Let k be a given positive integer. Then we have*

$$C_n(x_1, x_2, \ldots, x_{k+1}) = \sum \gamma(n; i_1, i_2, \ldots, i_n) e_{k-n+2}^{i_n} e_{k-n+3}^{i_{n-1}} \cdots e_k^{i_2} e_{k+1}^{i_1}, \tag{22}$$

where the summation is over all sequences $(i_1, i_2, \ldots, i_n)$ of nonnegative integers such that $i_1 + i_2 + \cdots + i_n = n$, $1 \leqslant i_1 \leqslant n-1$, $i_n = 0$ or $i_n = 1$. When $i_n = 1$, one has $i_1 = n-1$. The coefficients $\gamma(n; i_1, i_2, \ldots, i_n)$ equal the numbers of 0-1-2-$\cdots$-k-(k+1) increasing plane trees on $[n]$ with i_j degree $j-1$ vertices for all $1 \leqslant j \leqslant n$.

Proof. Let G_1 be the grammar given in Lemma 26. Consider a change of G_1. Note that $D_{G_1}(x_1) = e_{k+1},\ D_{G_1}(e_i) = (k-i+2)e_{i-1}e_{k+1}$ for $1 \leqslant i \leqslant k+1$. Thus we get a new grammar

$$G_2 = \{x_1 \to e_{k+1},\ e_i \to (k-i+2)e_{i-1}e_{k+1} \text{ for all } 1 \leqslant i \leqslant k+1\}. \tag{23}$$

Note that $D_{G_2}(x_1) = e_{k+1},\ D^2_{G_2}(x_1) = e_k e_{k+1},\ D^3_{G_2}(x_1) = e_k^2 e_{k+1} + 2e_{k-1}e_{k+1}^2,$

$$D^4_{G_2}(x_1) = e_k^3 e_{k+1} + 8e_{k-1}e_k e_{k+1}^2 + 6e_{k-2}e_{k+1}^3,$$
$$D^5_{G_2}(x_1) = e_k^4 e_{k+1} + 22e_k^2 e_{k-1}e_{k+1}^2 + 16e_{k-1}^2 e_{k+1}^3 + 42e_{k-2}e_k e_{k+1}^3 + 24e_{k-3}e_{k+1}^4.$$

By induction, we can assume that

$$D^n_{G_2}(x_1) = \sum \gamma(n; i_1, i_2, \ldots, i_n) e_{k-n+2}^{i_n} e_{k-n+3}^{i_{n-1}} \cdots e_k^{i_2} e_{k+1}^{i_1}. \tag{24}$$

We obtain

$$\begin{aligned} D^{n+1}_{G_2}(x_1) &= D_{G_2}\left(\sum \gamma(n; i_1, i_2, \ldots, i_n) e_{k-n+2}^{i_n} e_{k-n+3}^{i_{n-1}} \cdots e_k^{i_2} e_{k+1}^{i_1}\right) \\ &= \sum n i_n \gamma(n; i_1, i_2, \ldots, i_n) e_{k-n+1} e_{k-n+2}^{i_n-1} e_{k-n+3}^{i_{n-1}} \cdots e_k^{i_2} e_{k+1}^{i_1+1} + \\ &\sum (n-1) i_{n-1} \gamma(n; i_1, i_2, \ldots, i_n) e_{k-n+2}^{i_n+1} e_{k-n+3}^{i_{n-1}-1} \cdots e_k^{i_2} e_{k+1}^{i_1+1} + \cdots + \\ &\sum 2 i_2 \gamma(n; i_1, i_2, \ldots, i_n) e_{k-n+2}^{i_n} e_{k-n+3}^{i_{n-1}} \cdots e_{k-1}^{i_3+1} e_k^{i_2-1} e_{k+1}^{i_1+1} + \\ &\sum i_1 \gamma(n; i_1, i_2, \ldots, i_n) e_{k-n+2}^{i_n} e_{k-n+3}^{i_{n-1}} \cdots e_k^{i_2+1} e_{k+1}^{i_1}, \end{aligned}$$

which yields that the expansion (24) holds for $n+1$. Combining (24) and Lemma 26, we get (22). By induction, it is easy to verify $i_1 + i_2 + \cdots + i_n = n$, $1 \leqslant i_1 \leqslant n-1$, $i_n = 1$ or $i_n = 0$. In particular, when $i_n = 1$, one has $i_1 = n-1$.

Using (23), the combinatorial interpretation of $\gamma(n; i_1, i_2, \ldots, i_n)$ can be proved along the same lines as the proof of [8, Theorem 4.1]. However, we give a direct proof of it for our purpose. Let T be a 0-1-2-$\cdots$-k-(k+1) increasing plane tree on $[n]$. We first give a labeling of T as follows. Label a degree i vertex by e_{k-i+1} for all $0 \leqslant i \leqslant k+1$. In particular, label a leaf by e_{k+1} and label a degree $k+1$ vertex by 1. Let T' be a 0-1-2-$\cdots$-k-(k+1) increasing plane tree on $[n+1]$ by adding $n+1$ to T as a leaf. We can add $n+1$ to T only as a child of a vertex v that is not of degree $k+1$. For $1 \leqslant i \leqslant k+1$, if the vertex v is a degree $k-i+1$ vertex with the label e_i, there are $k-i+2$ cases to attach $n+1$ (from left to right, say). In either case, in T', the vertex v becomes a degree $k-i+2$ vertex with the label e_{i-1} and $n+1$ becomes a leaf with the label e_{k+1}. Hence the insertion of $n+1$ corresponds to the substitution rule $e_i \to (k-i+2)e_{i-1}e_{k+1}$. Hence $D^n_{G_2}(x_1)$ equals the sum of the weights of 0-1-2-$\cdots$-(k+1) increasing plane trees on $[n]$, and so we get the combinatorial interpretation of $\gamma(n; i_1, i_2, \ldots, i_n)$. $\square$

By using $D^{n+1}_{G_2}(x_1) = D_{G_2}\left(D^n_{G_2}(x_1)\right)$, it is routine to verify that

$$\gamma(n+1; 1, n, 0 \ldots, 0) = \gamma(n; 1, n-1, 0, \ldots, 0) = 1,$$
$$\gamma(n+1; n, 0, \ldots, 0, 1) = n\gamma(n; n-1, 0, \ldots, 0, 1) = n!.$$

Note that $\gamma(3;2,0,1)=2$, $\gamma(4;2,1,1,0)=8$ and

$$\gamma(n+1;2,n-2,1,0,\ldots,0)=2\gamma(n;2,n-3,1,0,\ldots,0)+2(n-1)\gamma(n;1,n-1,0,\ldots,0).$$

By induction, it is easy to verify that

$$\gamma(n;2,n-3,1,0,\ldots,0)=2^n-2n \text{ for } n\geqslant 3. \tag{25}$$

Let $C_n(x)=\sum_{j=1}^n C(n,j)x^j$, where the coefficients $C(n,j)$ are called the *second-order Eulerian numbers*. They satisfy the recursion

$$C_{n+1,j}=jC_{n,j}+(2n+2-j)C_{n,j-1},$$

with $C_{1,1}=1$ and $C_{1,j}=0$ if $j\neq 1$ (see [1]). In particular, $C_{n,2}=2^{n+1}-2(n+1)$. Comparing this with (25), we see that $\gamma(n;2,n-3,1,0,\ldots,0)=C_{n-1,2}$ for $n\geqslant 3$. Following Janson [18], the number $C_{n,j}$ equals the number of increasing plane trees on $[n+1]$ with j leaves. So we immediately get the following result.

Corollary 28. *For any $n\geqslant 2$ and $1\leqslant j\leqslant n-1$, one has*

$$C_{n-1,j}=\sum_{i_2+i_3+\cdots+i_n=n-j}\gamma(n;j,i_2,\ldots,i_{n-1},i_n).$$

7. Concluding remarks

In this paper, we introduce the SP-code of Stirling permutation, which is obtained from the well-known representation of Stirling permutations via ternary increasing trees. The SP-code can keep track of the parent node and the type of children (left, middle, right) of a ternary increasing tree. The concept of SP-code could be transferred to k-Stirling permutations [9,23,32], Stirling permutations of a general multiset [12,19,21] as well as quasi-Stirling permutations [13,33]. We plan to work out such generalizations in a separate contribution.

Declaration of competing interest

We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work, there is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of the manuscript.

Data availability

No data was used for the research described in the article.

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